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Stackelberg games and bilevel bilinear optimisation

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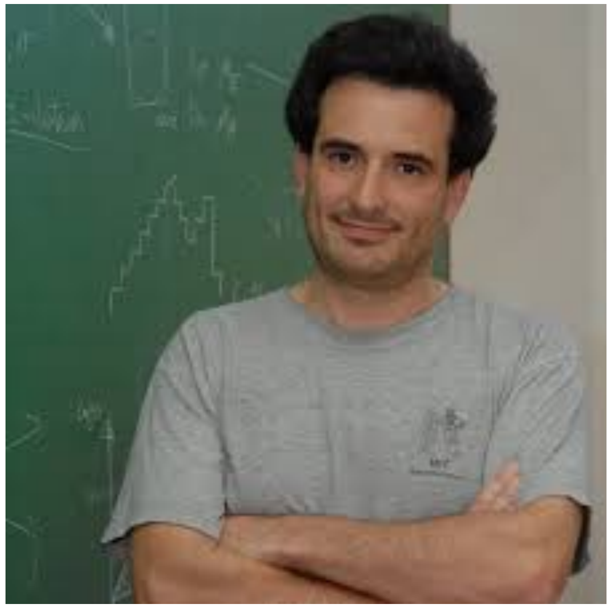

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My goals

- To show you a new and important domain of application of mathematics
- To introduce you to bilevel optimization
- To convince you that “to have a valid formulation” is not enough

Bilevel Problem

$$
\begin{array}{rl}
\max\limits_{x,y} & f(x,y) \\
\text{s.t.} & x \in X \\
 & y \in S(x) \\
\text{where} & S(x) = argmax_y g(x,y) \\
 & \text{s.t.}(x,y) \in Y
\end{array}
$$

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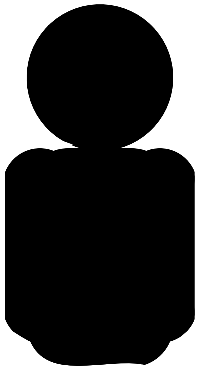
Adequate framework for Stackelberg game

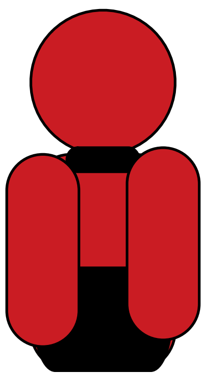
- Leader: 1st level,
- Follower: 2nd level,
- Leader takes follower's optimal reaction into account.

(1905 - 1946)

Bimatrix game



Follower



Leader

	C	D	
A	(2,1)	(4,0)	0.5
B	(1,0)	(3,2)	0.5

Leader Mixed Strategy

Follower Pure Strategy

A Stackelberg solution to the game $\longrightarrow$ (B,D) yielding a payoff of (3.5,1)

Stackelberg vs Nash

	Player 2 - C	**Player 2 - D**
Player 1 - A	***(2,1)***	***(4,0)***
Player 1 - B	(1,0)	(3,2)

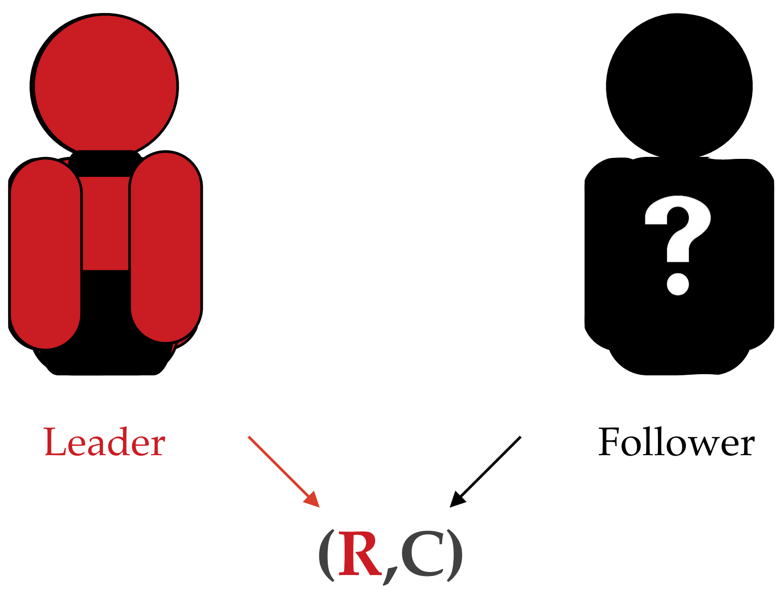
Nash equilibrium: Player 1-A and Player 2-C => (2,1)

Stackelberg solution: Player 1-B and Player 2-D => (3,2)

Nash equilibrium may not exist
There is always a Stackelberg solution (optimistic)

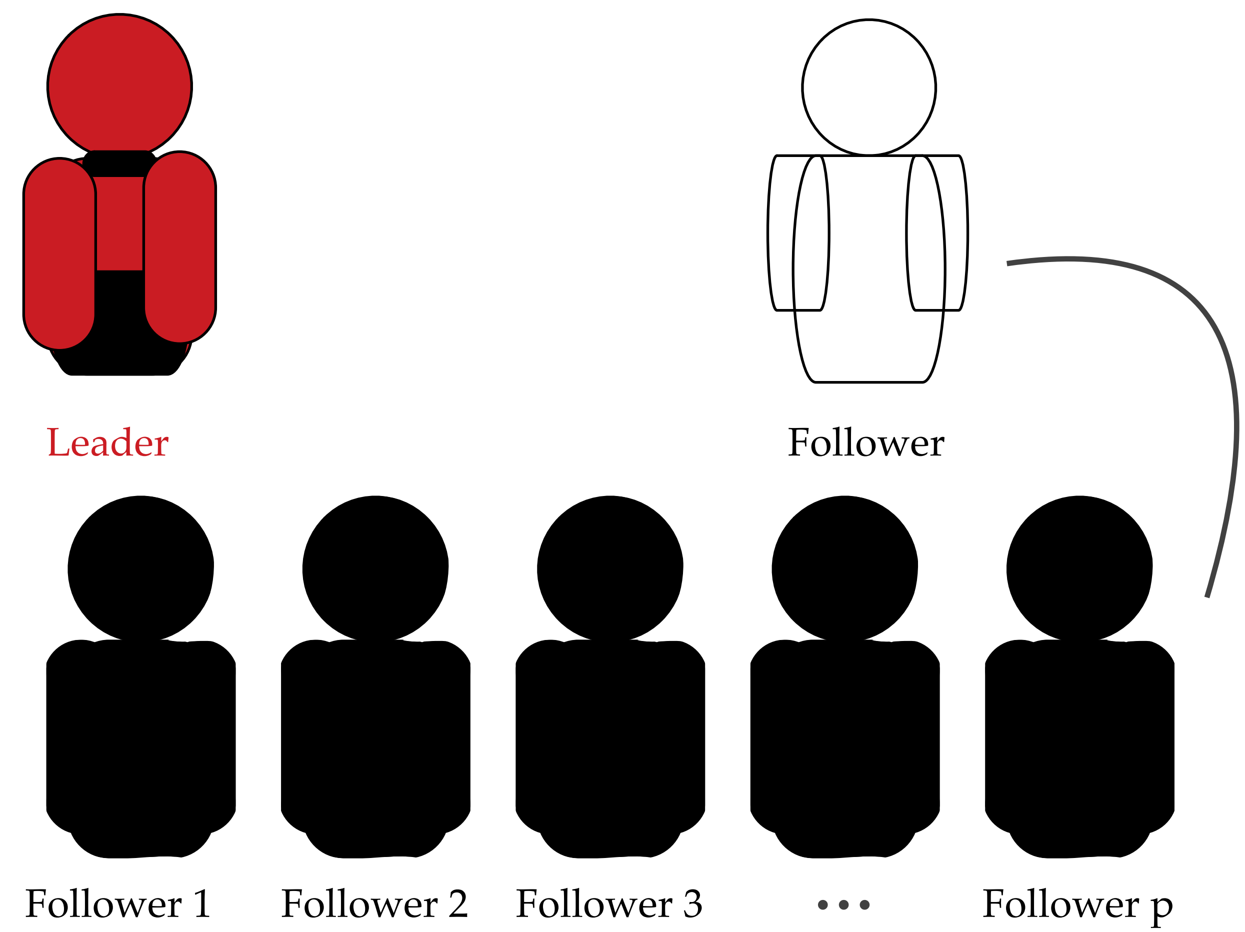
Stackelberg Games

Stackelberg Game



p-Followers Stackelberg Game

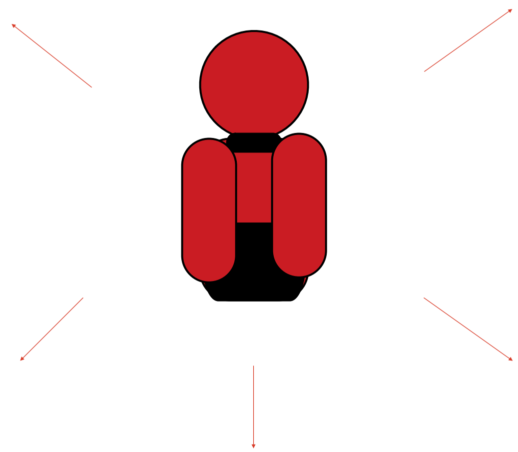
(Conitzer and Sandblom, 2006)

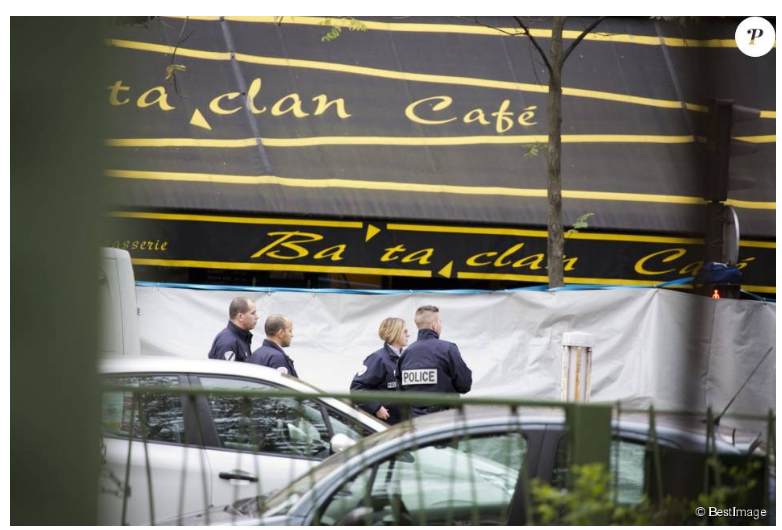


Objective of the Game

- Reward-maximizing strategy for the Leader.
- Follower will best respond to observable Leader's strategy.

Applications
(Tambe et al., USC)





The beauty of this approach

comes from

the randomisation

1-Follower general Stackelberg game

- Follower optimally chooses one strategy j with probability 1
- For each possible strategy j of the follower, determine the probabilities x_i that leader chooses strategy i by solving the LP:

$$\begin{aligned} \max \quad & \sum_{i \in I} R_{ij} x_i \\ \text{s.t.} \quad & \sum_{i \in I} x_i = 1 \\ & x_i \geq 0 \\ & \sum_{i \in I} C_{ij} x_i \geq \sum_{i \in I} C_{il} x_i, \forall l \in J \end{aligned}$$

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Modeling a p-followers general Stackelberg game

Follower type $k \in K$ and $\pi \in [0,1]$

$$R^k, C^k \in \mathbb{R}^{|I| \times |J|}, \ \forall k \in K$$

$$x \in \mathbb{S}^{|I|} := \{x \in [0,1]^{|I|} : \sum_{i \in I} x_i = 1\}$$

$x_i =$ probability with which the Leader plays pure strategy i

$$q^k \in \mathbb{S}^{|J|} := \{q \in [0,1]^{|J|} : \sum_{j \in J} q_j = 1\}, \ \forall k \in K$$

$q^k_j =$ probability with which type k Follower plays pure strategy j

Bilevel formulation

$$
\begin{aligned}
(\text{BIL-}p\text{-G}) \quad \text{Max}_{x,q} \quad & \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \pi^k R_{ij}^k x_i q_j^k \\
\text{s.t.} \quad & \sum_{i\in I} x_i = 1, \\
& x_i \in [0,1] && \forall i \in I, \\
& q^k = \arg\max_{r^k} \left\{ \sum_{i\in I}\sum_{j\in J} C_{ij}^k x_i r_j^k \right\} && \forall k \in K, \\
& r_j^k \in [0,1] && \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} r_j^k = 1 && \forall k \in K.
\end{aligned}
$$

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Bilinear formulation

Paruchuri et al.(2008)

$$
\begin{aligned}
\text{(QUAD)} \quad & \max_{x,q,a} && \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \pi^k R_{ij}^k x_i q_j^k && \\
& \text{s.t.} && \sum_{i\in I} x_i = 1, && \\
& && \sum_{j\in J} q_j^k = 1 && \forall k \in K, \\
& && 0 \le (a^k - \sum_{i\in I} C_{ij}^k x_i) \le (1 - q_j^k)M && \forall j \in J, \forall k \in K, \\
& && x_i \in [0,1] && \forall i \in I, \\
& && q_j^k \in \{0,1\} && \forall j \in J, \forall k \in K, \\
& && a^k \in \mathbb{R} && \forall k \in K.
\end{aligned}
$$

MIP1 Kiekintvelt et al. (2008)

$$
\begin{aligned}
\text{(MIP1)} \quad & \max_{x,q,a,d} && \sum_{k \in K} \pi^k d^k && \\
& \text{s.t.} && d^k \leq \sum_{i \in I} R^k_{i,j} x_i + M_1(1 - q^k_j), && \forall j \in J, \forall k \in K, \\
& && \sum_{i \in I} x_i = 1, && \\
& && \sum_{j \in J} q^k_j = 1 && \forall k \in K, \\
& && 0 \leq (a^k - \sum_{i \in I} C^k_{ij} x_i) \leq M_2(1 - q^k_j) && \forall j \in J, \forall k \in K, \\
& && x_i \in [0, 1] && \forall i \in I, \\
& && q^k_j \in \{0, 1\} && \forall j \in J, \forall k \in K, \\
& && a^k \in \mathbb{R} && \forall k \in K.
\end{aligned}
$$

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Linearize

$$x_i q_j^k = z_{ij}^k, \forall i \in I, j \in J, k \in K$$

- $z_{ij}^k \in [0,1], \forall i \in I, j \in J, k \in K$
- $x_i = \sum_{j \in J} z_{ij}^k, \forall i \in I, k \in K$
- $q_j^k = \sum_{i \in I} z_{ij}^k, \forall j \in J$

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MIP2 Paruchuri (2008)

$$
\begin{aligned}
\text{(MIP2)} \quad & \max_{x,q,a} && \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \pi^k R_{ij}^k z_{ij}^k \\
\text{s.t.} \quad & && x_i = \sum_{j\in J} z_{ij}^k, && \forall i \in I, k \in K \\
& && q_j^k = \sum_{i\in I} z_{ij}^k, && \forall j \in J \\
& && \sum_{i\in I} x_i = 1, \\
& && \sum_{j\in J} q_j^k = 1 && \forall k \in K, \\
& && 0 \leq (a^k - \sum_{i\in I} C_{ij}^k x_i) \leq (1 - q_j^k)M && \forall j \in J, \forall k \in K, \\
& && z_{ij}^k \in [0,1] && \forall i \in I, \forall j \in J, \forall k \in K, \\
& && x_i \in [0,1] && \forall i \in I, \\
& && q_j^k \in \{0,1\} && \forall j \in J, \forall k \in K, \\
& && a^k \in \mathbb{R} && \forall k \in K.
\end{aligned}
$$

Eliminate a^k

$$0 \leq (a^k - \sum_{i \in I} C_{ij}^k x_i) \leq (1 - q_j^k) M, \forall j \in J, \forall k \in K$$

$$\sum_{i \in I} C_{ij}^k x_i \leq a^k \leq \sum_{i \in I} C_{il}^k x_i + M(1 - q_l^k),$$

$$\forall j, l \in J, k \in K$$

$$\sum_{i \in I} (C_{il}^k - C_{ij}^k) x_i \leq (1 - q_j^k) M, \forall j, l \in J, \forall k \in K$$

Apply RLT

Sheraly, Adams (1999)

$$\boxed{\sum_{i\in I}(C_{il}^k - C_{ij}^k)x_i \leq (1-q_j^k)M, \forall j,l \in J, \forall k \in K}$$

$$\Downarrow$$

$$\boxed{\sum_{i\in I}(C_{il}^k - C_{ij}^k)x_i q_j^k \leq (1-q_j^k)q_j^k M}$$

$$\Downarrow$$

$$\boxed{\sum_{i\in I}(C_{il}^k - C_{ij}^k)z_{ij}^k \leq 0, \forall j,l \in J, \forall k \in K}$$

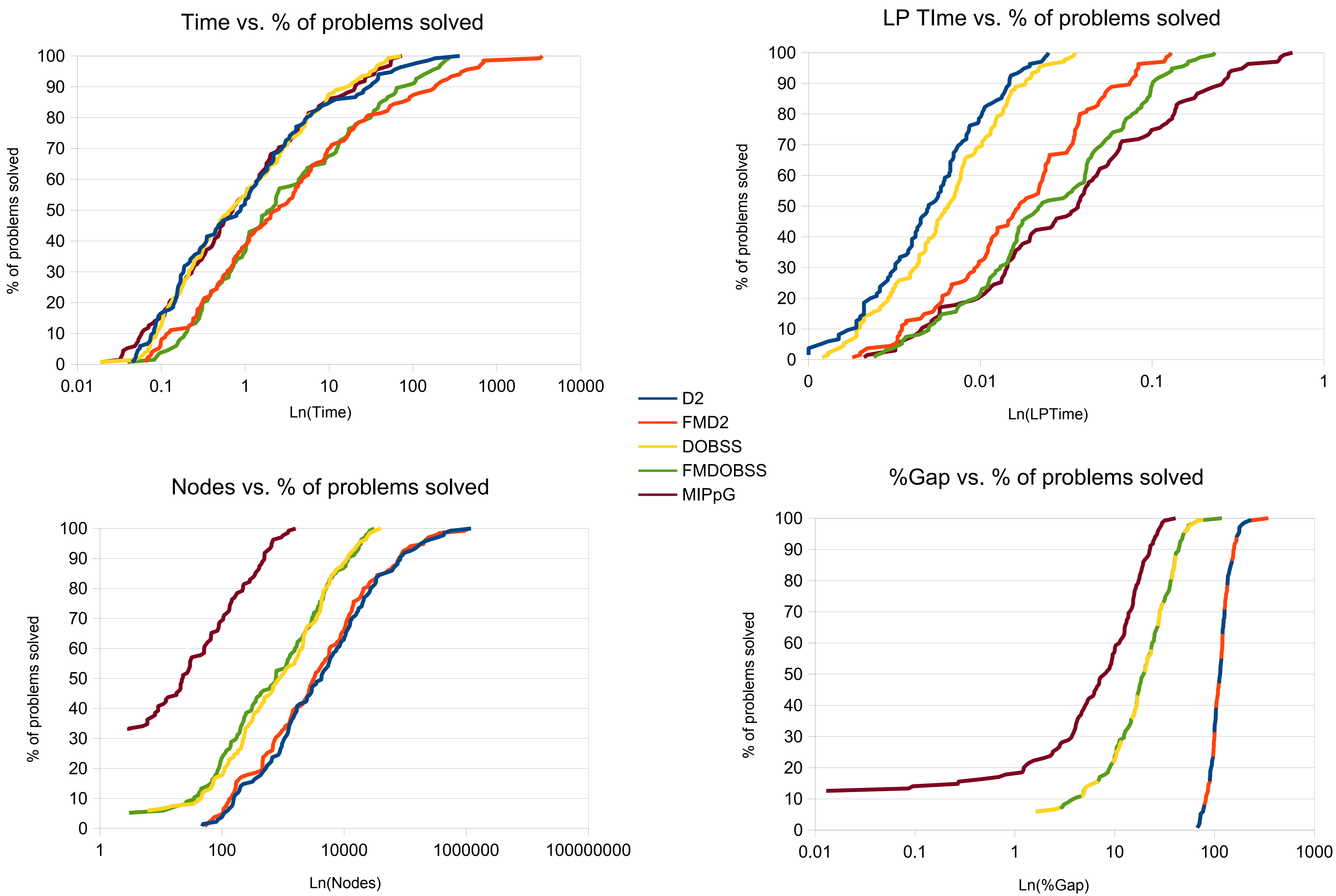
MIP3

$$
\begin{array}{llll}
\text{(MIP-}p\text{-G)} & \max\limits_{x,q} & \displaystyle\sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\pi^k R_{ij}^k z_{ij}^k & \\
& \text{s.t.} & \displaystyle\sum_{i\in I}\sum_{j\in J} z_{ij}^k = 1, & \forall k \in K \\
& & \displaystyle\sum_{i\in I}(C_{ij}^k - C_{i\ell}^k) z_{ij}^k \geq 0 & \forall j,\ell \in J, \forall k \in K, \\
& & z_{ij}^k \geq 0 & \forall i \in I, \forall j \in J, \forall k \in K, \\
& & \displaystyle\sum_{i\in I} z_{ij}^k \in \{0,1\} & \forall j \in J, \forall k \in K, \\
& & \displaystyle\sum_{j\in J} z_{ij}^k = \sum_{j\in J} z_{ij}^1 & \forall i \in I, \forall k \in K.
\end{array}
$$

About those MIPs

- $z_{LP}(MIP3) \leq z_{LP}(MIP2) \leq z_{LP}(MIP1)$
- $LP(MIP3)$ is "integer" for $k = 1$

Computational comparison



GSGs: I={10,20,30}, J={10,20,30}, K={2,4,6}–without variability.

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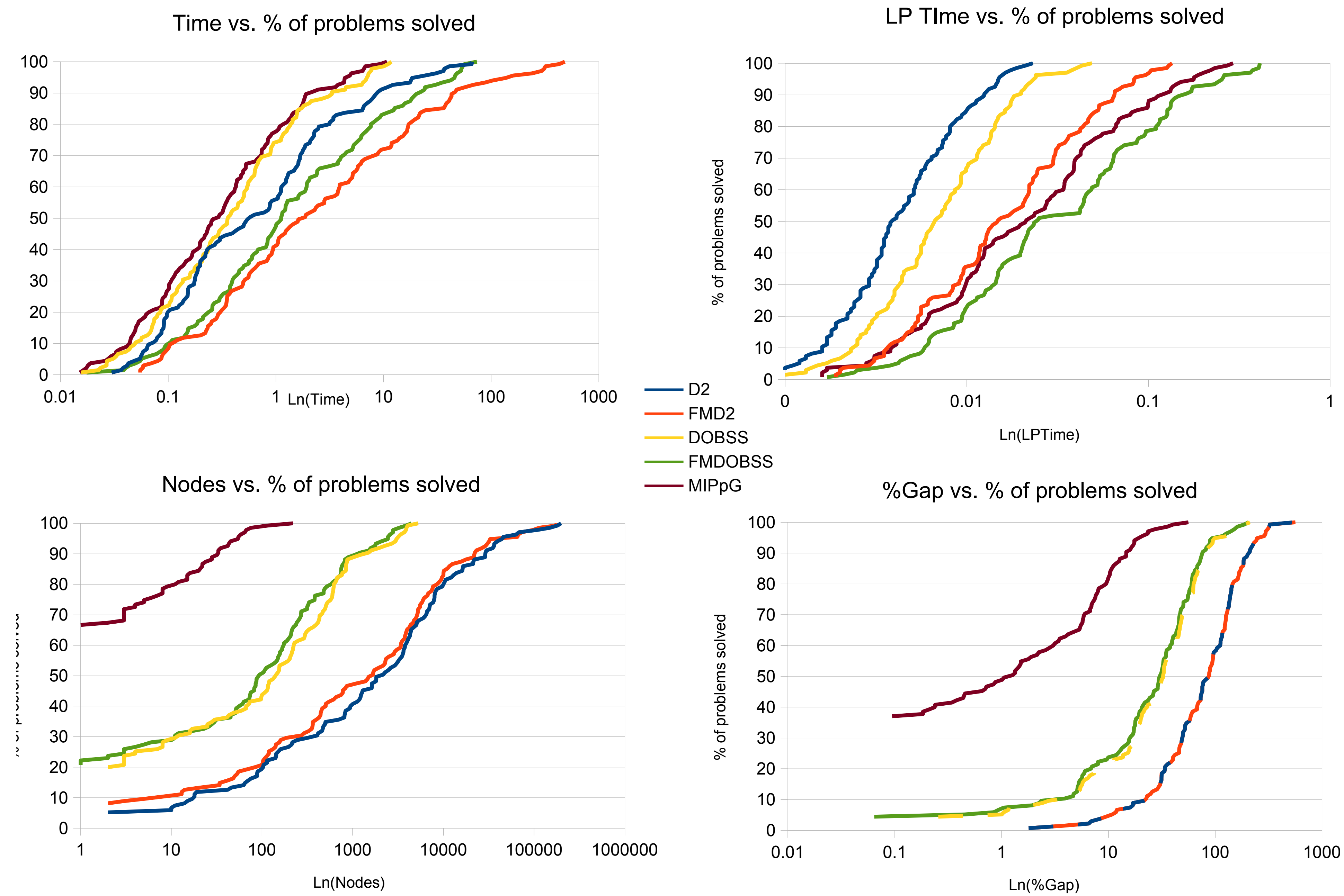
Computational comparison



GSGs: I={10,20,30}, J={10,20,30}, K={2,4,6}–with variability.

	(D2)	(FMD2)	(DOBSS)	(FMDOBSS)	(MIP-p-G)
Mean Gap %	110.56	110.56	31.88	30.64	7.56

Stackelberg security game

- Payoffs depend only on which target is attacked and whether it is covered or not

		Covered	Uncovered
•	Defender	$D^k(j\|c)$	$D^k(j\|u)$
	Attacker	$A^k(j\|c)$	$A^k(j\|u)$

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Compact representation of Stackelberg Security Games

- Resources-Targets settings can be modeled as a Stackelberg Game BUT if m ressources and n targets then $\binom{n}{m}$ pure strategies!
- Stackelberg Security Games can be more compactly represented.
- Solve for optimal coverage probabilities of the targets.

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Stackelberg security game:

"extended formulation"

$$
\begin{aligned}
\text{(QUAD)} \quad \max_{x,q,a} \quad & \sum_{k\in K} \pi^k \sum_{j\in J} q_j^k \big(D^k(j|c) \sum_{i\in I: j\in i} x_i + D^k(j|u) \sum_{i\in I: j\notin i} x_i \big) && \\
\text{s.t.} \quad & \sum_{i\in I} x_i = 1, && \\
& \sum_{j\in J} q_j^k = 1 && \forall k \in K, \\
& 0 \le a^k - \big(A^k(j|c) \sum_{i\in I: j\in i} x_i + A^k(j|u) \sum_{i\in I: j\notin i} x_i \big) \le (1 - q_j^k) M && \forall j \in J, \forall k \in K, \\
& x_i \in [0,1] && \forall i \in I, \\
& q_j^k \in \{0,1\} && \forall j \in J, \forall k \in K, \\
& a^k \in \mathbb{R} && \forall k \in K.
\end{aligned}
$$

Stackelberg security game:

“extended formulation”

$$
\begin{aligned}
\text{(QUAD)} \quad & \max_{x,q,a} && \sum_{k \in K} \pi^k \sum_{j \in J} q_j^k \Big(D^k(j|c) \underbrace{\boxed{\sum_{i \in I : j \in i} x_i}}_{c_j} + D^k(j|u) \underbrace{\boxed{\sum_{i \in I : j \notin i} x_i}}_{1 - c_j} \Big) \\
& \text{s.t.} && \sum_{i \in I} x_i = 1, \\
& && \sum_{j \in J} q_j^k = 1 && \forall k \in K, \\
& && 0 \leq a^k - \Big(A^k(j|c) \underbrace{\boxed{\sum_{i \in I : j \in i} x_i}}_{c_j} + A^k(j|u) \underbrace{\boxed{\sum_{i \in I : j \notin i} x_i}}_{1 - c_j} \Big) \leq (1 - q_j^k) M && \forall j \in J, \forall k \in K, \\
& && x_i \in [0, 1] && \forall i \in I, \\
& && q_j^k \in \{0, 1\} && \forall j \in J, \forall k \in K, \\
& && a^k \in \mathbb{R} && \forall k \in K.
\end{aligned}
$$

Stackelberg security game:

compact formulation

$$
\begin{aligned}
\text{(QUAD)} \quad \max_{x,q,a} \quad & \sum_{k\in K} \pi^k \sum_{j\in J} q_j^k (D^k(j|c)c_j + D^k(j|u)(1-c_j)) \\
\text{s.t.} \quad & \boxed{\begin{aligned} & \sum_{i\in I} x_i = 1, \\ & \sum_{i:j\in i} x_i = c_j & & \forall j \in J, \\ & x_i \in [0,1] & & \forall i \in I, \end{aligned}} \\
& \sum_{j\in J} q_j^k = 1 & & \forall k \in K, \\
& 0 \le a^k - (A^k(j|c)c_j + A^k(j|u)(1-c_j)) \le (1-q_j^k)M & & \forall j \in J, \forall k \in K, \\
& q_j^k \in \{0,1\} & & \forall j \in J, \forall k \in K, \\
& a^k \in \mathbb{R} & & \forall k \in K.
\end{aligned}
$$

Stackelberg security game

compact formulation

$$
\begin{aligned}
\text{(SECU-K-Quad)} \quad \text{Max}_c \quad & \sum_{k \in K} \sum_{j \in J} p_k (q_j^k (c_j D^k(j|c) + (1 - c_j) D^k(j|u))) & \\
\text{s.t.} \quad & \boxed{\begin{array}{l} c_j \in [0,1] \\ \displaystyle\sum_{j \in J} c_j \le m, \end{array}} & \forall j \in J \\
& q_j^k (c_j A^k(j|c) - (1 - c_j) A^k(j|u)) \ge q_j^k (c_t A^k(t|c) - (1 - c_t) A^k(t|u)) & \forall k \in K, \\
& q_j^k \in \{0, 1\} & \forall j \in J, \forall k \in K \\
& \sum_{j \in J} q_j^k = 1 & \forall k \in K.
\end{aligned}
$$

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Stackelberg security game

Extended formulations (x-space): $\boxed{\text{MIP1}} \prec \boxed{\text{MIP2}} \prec \boxed{\text{MIP3}}$

$$= \qquad \not\subset \qquad \not\subset$$

Compact formulations (c-space): $\boxed{\text{MIP1}} \prec \boxed{\text{MIP2}} \prec \boxed{\text{MIP3}}$

Stackelberg security game: MIP3-compact

$$
\begin{aligned}
\text{(SECU-p-MIP)} \quad \text{Max}_y \quad & \sum_{k \in K} \sum_{j \in J} p_k (D^k(j|c) y^k_{jj} + D^k(j|u)(q^k_j - y^k_{jj})) & \\
\text{s.t.} \quad & \sum_{l \in J} y^k_{lj} \leq m q^k_j & \forall k, j, \\
& 0 \leq y^k_{lj} \leq q^k_j, & \forall k, j \\
& \sum_{j \in J} q^k_j = 1, & \forall k, \\
& A^k(j|c) y^k_{jj} + A^k(j|u)(q^k_j - y^k_{jj}) - A(l|c) y^k_{lj} & \\
& - A(l|u)(q^k_l - y^k_{lj}) \geq 0 & \forall j, l, k, \\
& \sum_{l \in J} y^k_{lj} \in \{0, 1\} & \forall l, k, \\
& \sum_{j \in J} y^k_{lj} = \sum_{j \in J} y^1_{lj} & \forall l, k.
\end{aligned}
$$

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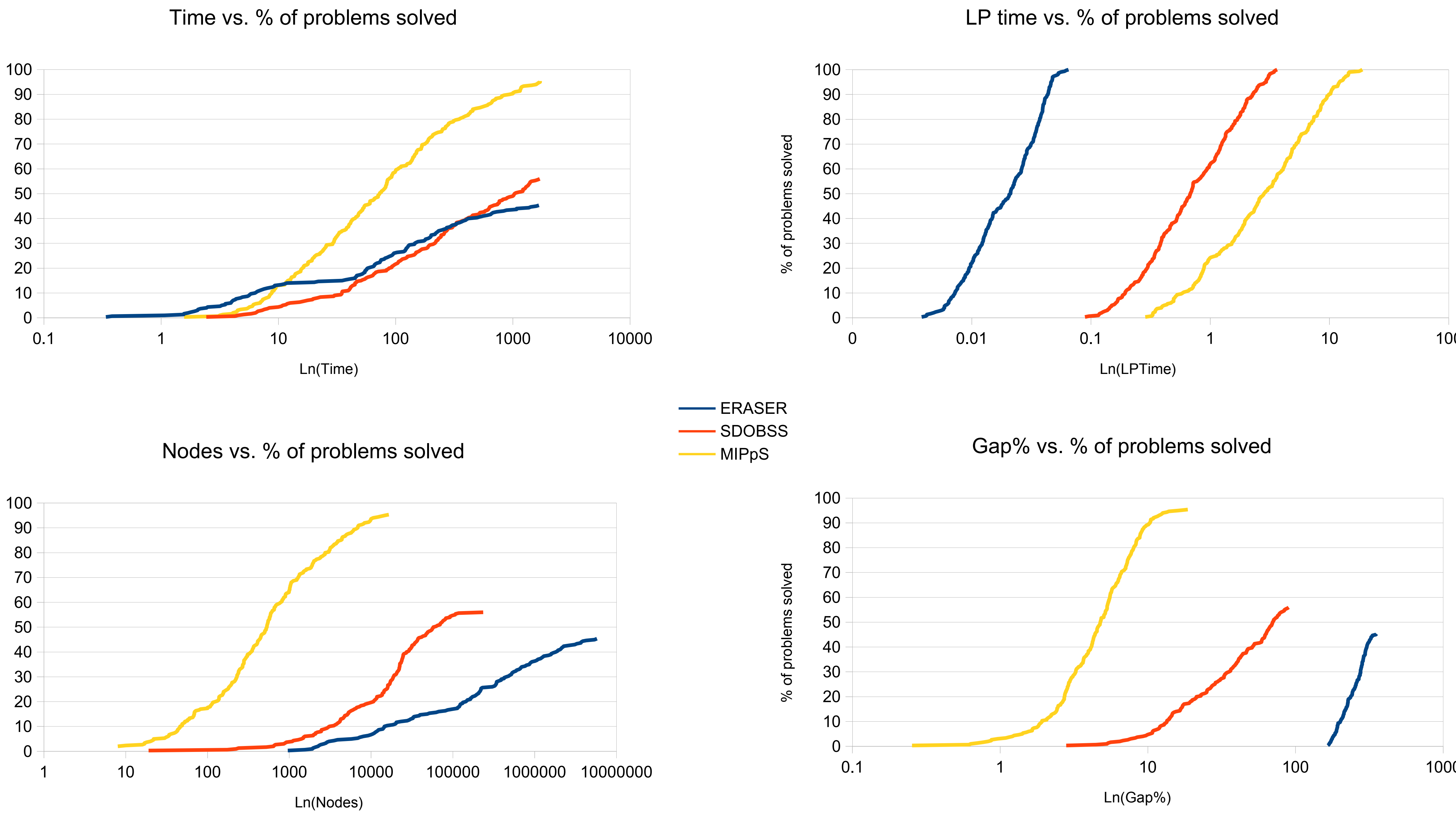
Link between MIP3 and SECU-p-MIP

- A pure strategy of the defender is a set of at most m targets
- $y_{hj}^k = \sum_{i \in I : h \in i} z_{ij}^k$
- $Proj(LP(P_{MIP3})) \subset LP(P_{SECU-p-MIP})$



$$\text{SSGs: } |K| \in \{4,6,8,12\}, |J| \in \{30,40,50,60,70\}, m \in \{0.25|J|, 0.50|J|, 0.75|J|\}$$

	(ERASER)	(SDBOSS)	(MIP-p-S)
Mean Gap %	204.82	28.76	1.72

Conclusions

- Bilevel models and MIP reformulations are appropriate to solve Stackelberg bimatrix games
- New MIP formulations for general and security cases
- A valid formulation is not enough!
- Future: develop decomposition solution approach (DW, Benders) based on strongest model.
- Future: study problems with non homogeneous ressources, different second level…


THANK YOU


Questions?

How to determine mixed strategies from coverage probabilities

$$
\begin{aligned}
\sum_{h:j\in h} x_h &= c_j, \forall j \\
\sum_{h} x_h &= 1 \\
x_h &\geq 0, \forall h
\end{aligned}
$$

Example:m=2

C_1=0.7, C_2=0.8, C_3=0.5