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EURO Journal on Computational Optimization – 2014 Publisher's Report

Stackelberg games and bilevel bilinear optimisation

MODELO DE STACKELBERG (Leader – Follower)





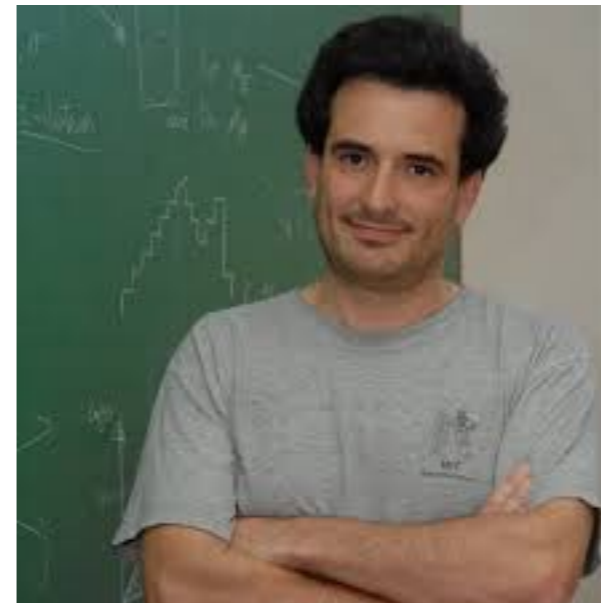
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My goals

- To show you a new and important domain of application of mathematics
- To introduce you to bilevel optimization
- To convince you that “to have a valid formulation” is not enough

Bilevel Problem

$$\max_{x,y} f(x,y)$$

$$\text{s.t. } x \in X$$

$$y \in S(x)$$

$$\text{where } S(x) = \operatorname{argmax}_y g(x,y)$$

$$\text{s.t. } (x,y) \in Y$$

Adequate framework for Stackelberg game

- Leader: 1st level,
- Follower: 2nd level,
- Leader takes follower's optimal reaction into account.



**Heinrich
von Stackelberg**
(1905 - 1946)

Bimatrix game

		Follower		
		C	D	
Leader	A	(2,1)	(4,0)	← 0.5
	B	(1,0)	(3,2)	← 0.5

↑
Follower
Pure Strategy

Leader
Mixed
Strategy

A Stackelberg solution to the game → (B,D) yielding a payoff of (3.5,1)

Stackelberg vs Nash

	Player 2 - C	Player 2 - D
Player 1 - A	(2,1)	(4,0)
Player 1 - B	(1,0)	(3,2)

Nash equilibrium: Player 1-A and Player 2-C => (2,1)

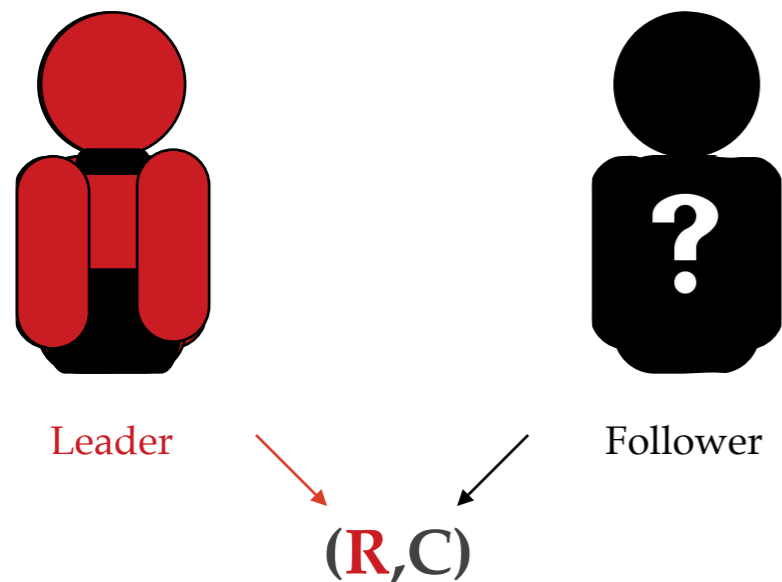
Stackelberg solution: Player 1-B and Player 2-D => (3,2)

Nash equilibrium may not exist

There is always a Stackelberg solution (optimistic)

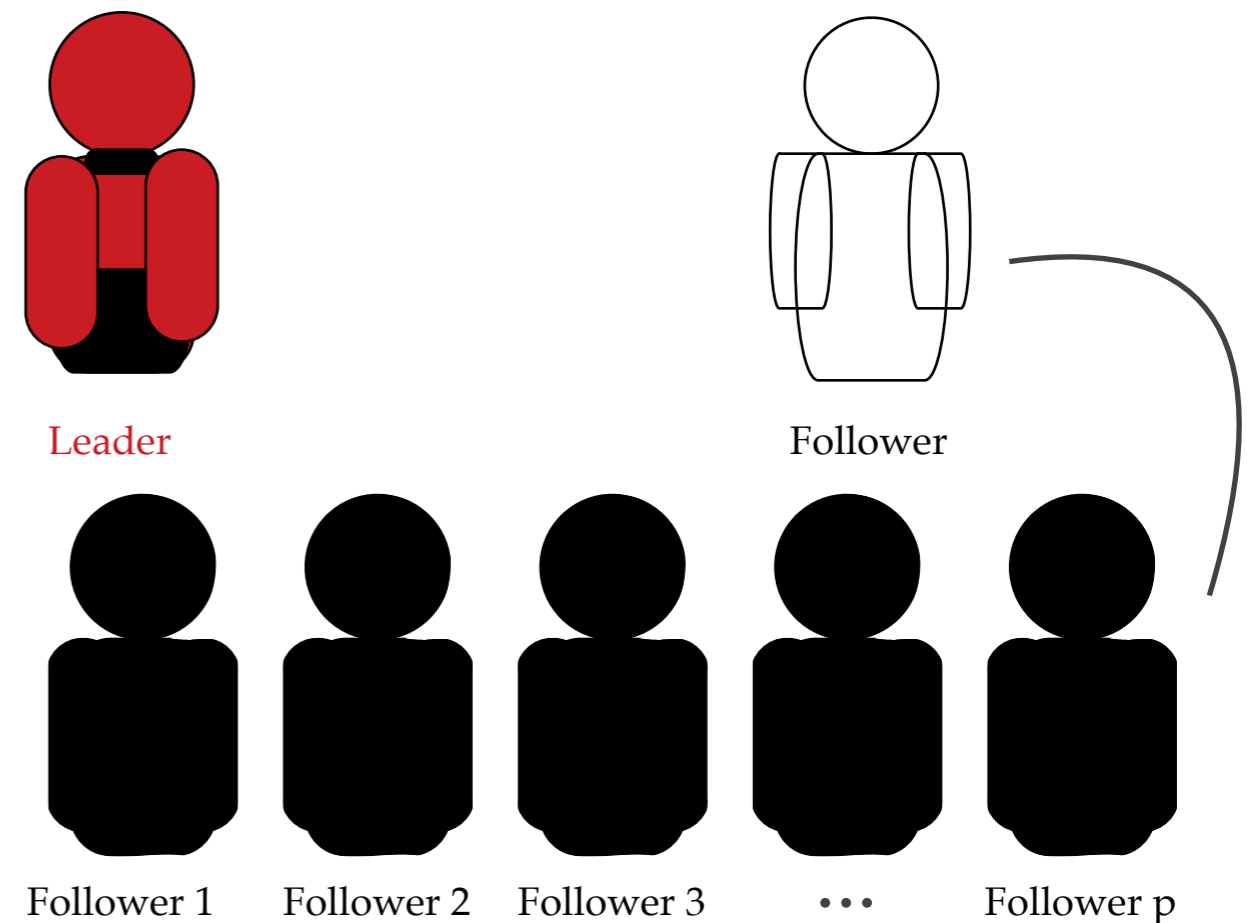
Stackelberg Games

Stackelberg Game



p-Followers Stackelberg Game

(Conitzer and Sandblom, 2006)

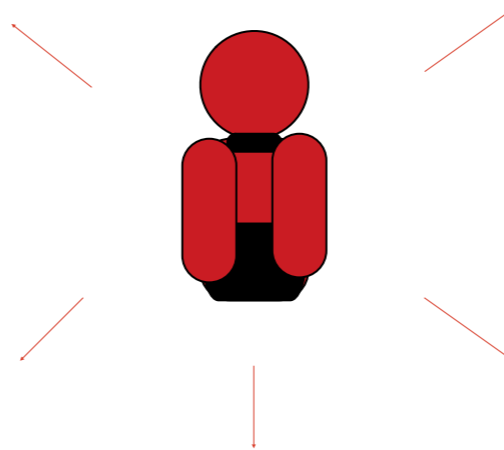


Objective of the Game

- Reward-maximizing strategy for the Leader.
- Follower will best respond to observable Leader's strategy.

Applications

(Tambe et al., USC)



The beauty of this approach
comes from
the randomisation

1-Follower general Stackelberg game

- Follower optimally chooses one strategy j with probability 1
- For each possible strategy j of the follower, determine the probabilities x_i that leader chooses strategy i by solving the LP:

$$\begin{aligned} \max \quad & \sum_{i \in I} R_{ij} x_i \\ \text{s.t.} \quad & \sum_{i \in I} x_i = 1 \\ & x_i \geq 0 \\ & \sum_{i \in I} C_{ij} x_i \geq \sum_{i \in I} C_{il} x_i, \forall l \in J \end{aligned}$$

Modeling a p-followers general Stackelberg game

Follower type $k \in K$ and $\pi \in [0, 1]$

$$R^k, C^k \in \mathbb{R}^{|I| \times |J|}, \forall k \in K$$

$$x \in \mathbb{S}^{|I|} := \{x \in [0, 1]^{|I|} : \sum_{i \in I} x_i = 1\}$$

x_i = probability with which the Leader plays pure strategy i

$$q^k \in \mathbb{S}^{|J|} := \{q \in [0, 1]^{|J|} : \sum_{j \in J} q_j = 1\}, \forall k \in K$$

q_j^k = probability with which type k Follower plays pure strategy j

Bilevel formulation

$$\begin{aligned}
 \text{(BIL-}p\text{-G)} \quad & \text{Max}_{x,q} \quad \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \pi^k R_{ij}^k x_i q_j^k \\
 \text{s.t.} \quad & \sum_{i \in I} x_i = 1, \\
 & x_i \in [0, 1] \quad \forall i \in I, \\
 & q^k = \arg \max_{r^k} \left\{ \sum_{i \in I} \sum_{j \in J} C_{ij}^k x_i r_j^k \right\} \quad \forall k \in K, \\
 & r_j^k \in [0, 1] \quad \forall j \in J, \forall k \in K, \\
 & \sum_{j \in J} r_j^k = 1 \quad \forall k \in K.
 \end{aligned}$$

Bilinear formulation

Paruchuri et al.(2008)

$$\begin{aligned} \text{(QUAD)} \quad & \max_{x,q,a} \quad \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \pi^k R_{ij}^k x_i q_j^k \\ \text{s.t.} \quad & \sum_{i \in I} x_i = 1, \\ & \sum_{j \in J} q_j^k = 1 \quad \forall k \in K, \\ & 0 \leq (a^k - \sum_{i \in I} C_{ij}^k x_i) \leq (1 - q_j^k)M \quad \forall j \in J, \forall k \in K, \\ & x_i \in [0, 1] \quad \forall i \in I, \\ & q_j^k \in \{0, 1\} \quad \forall j \in J, \forall k \in K, \\ & a^k \in \mathbb{R} \quad \forall k \in K. \end{aligned}$$

MIP1 Kiekintvelt et al. (2008)

$$\begin{aligned} \text{(MIP1)} \quad & \max_{x,q,a,d} \quad \sum_{k \in K} \pi^k d^k \\ \text{s.t.} \quad & d^k \leq \sum_{i \in I} R_{i,j}^k x_i + M_1(1 - q_j^k), & \forall j \in J, \forall k \in K, \\ & \sum_{i \in I} x_i = 1, \\ & \sum_{j \in J} q_j^k = 1 & \forall k \in K, \\ & 0 \leq (a^k - \sum_{i \in I} C_{ij}^k x_i) \leq M_2(1 - q_j^k) & \forall j \in J, \forall k \in K, \\ & x_i \in [0, 1] & \forall i \in I, \\ & q_j^k \in \{0, 1\} & \forall j \in J, \forall k \in K, \\ & a^k \in \mathbb{R} & \forall k \in K. \end{aligned}$$

Linearize

$$x_i q_j^k = z_{ij}^k, \forall i \in I, j \in J, k \in K$$



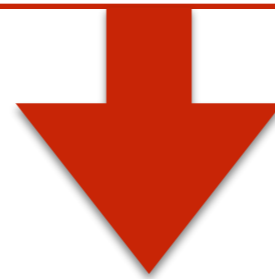
- $z_{ij}^k \in [0, 1], \forall i \in I, j \in J, k \in K$
- $x_i = \sum_{j \in J} z_{ij}^k, \forall i \in I, k \in K$
- $q_j^k = \sum_{i \in I} z_{ij}^k, \forall j \in J$

MIP2 Paruchuri (2008)

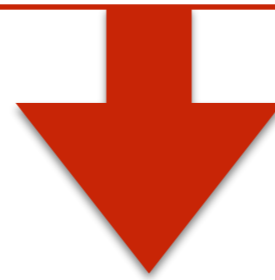
$$\begin{aligned}
 \text{(MIP2)} \quad & \max_{x,q,a} \quad \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \pi^k R_{ij}^k z_{ij}^k \\
 \text{s.t.} \quad & x_i = \sum_{j \in J} z_{ij}^k, & \forall i \in I, k \in K \\
 & q_j^k = \sum_{i \in I} z_{ij}^k, & \forall j \in J \\
 & \sum_{i \in I} x_i = 1, \\
 & \sum_{j \in J} q_j^k = 1 & \forall k \in K, \\
 & 0 \leq (a^k - \sum_{i \in I} C_{ij}^k x_i) \leq (1 - q_j^k)M & \forall j \in J, \forall k \in K, \\
 & z_{ij}^k \in [0, 1] & \forall i \in I, \forall j \in J, \forall k \in K, \\
 & x_i \in [0, 1] & \forall i \in I, \\
 & q_j^k \in \{0, 1\} & \forall j \in J, \forall k \in K, \\
 & a^k \in \mathbb{R} & \forall k \in K.
 \end{aligned}$$

Eliminate a^k

$$0 \leq (a^k - \sum_{i \in I} C_{ij}^k x_i) \leq (1 - q_j^k)M, \forall j \in J, \forall k \in K$$



$$\sum_{i \in I} C_{ij}^k x_i \leq a^k \leq \sum_{i \in I} C_{il}^k x_i + M(1 - q_l^k),$$
$$\forall j, l \in J, k \in K$$

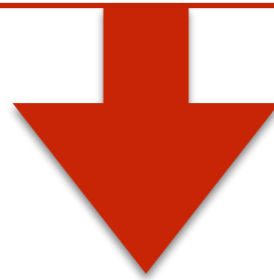


$$\sum_{i \in I} (C_{il}^k - C_{ij}^k) x_i \leq (1 - q_j^k)M, \forall j, l \in J, \forall k \in K$$

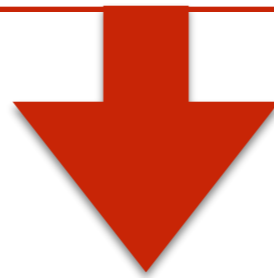
Apply RLT

Sheraly, Adams (1999)

$$\sum_{i \in I} (C_{il}^k - C_{ij}^k) x_i \leq (1 - q_j^k) M, \forall j, l \in J, \forall k \in K$$



$$\sum_{i \in I} (C_{il}^k - C_{ij}^k) x_i q_j^k \leq (1 - q_j^k) q_j^k M$$



$$\sum_{i \in I} (C_{il}^k - C_{ij}^k) z_{ij}^k \leq 0, \forall j, l \in J, \forall k \in K$$

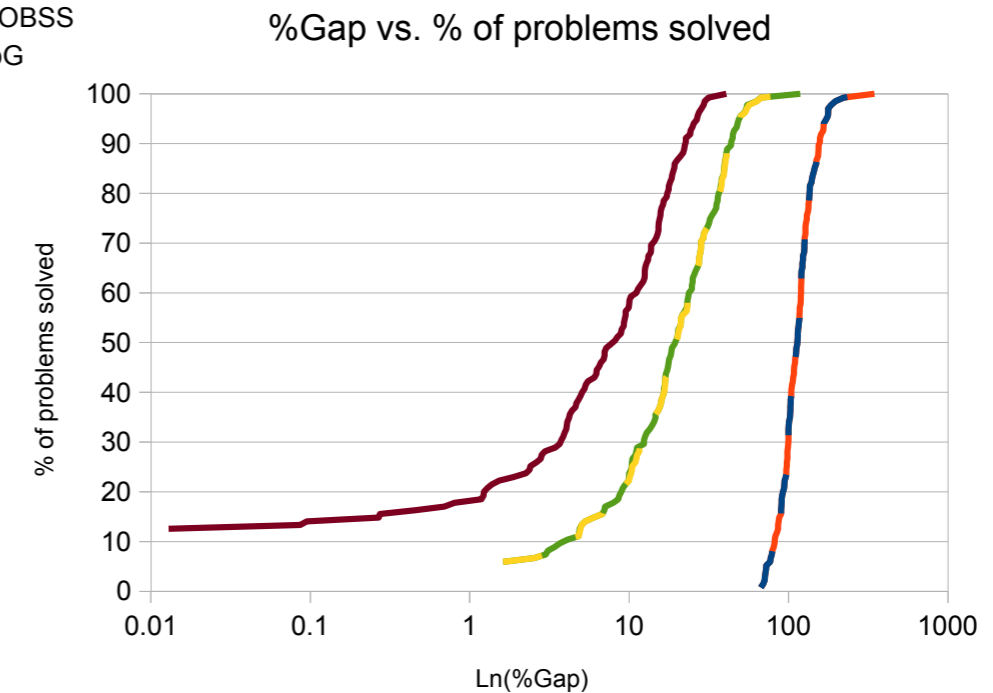
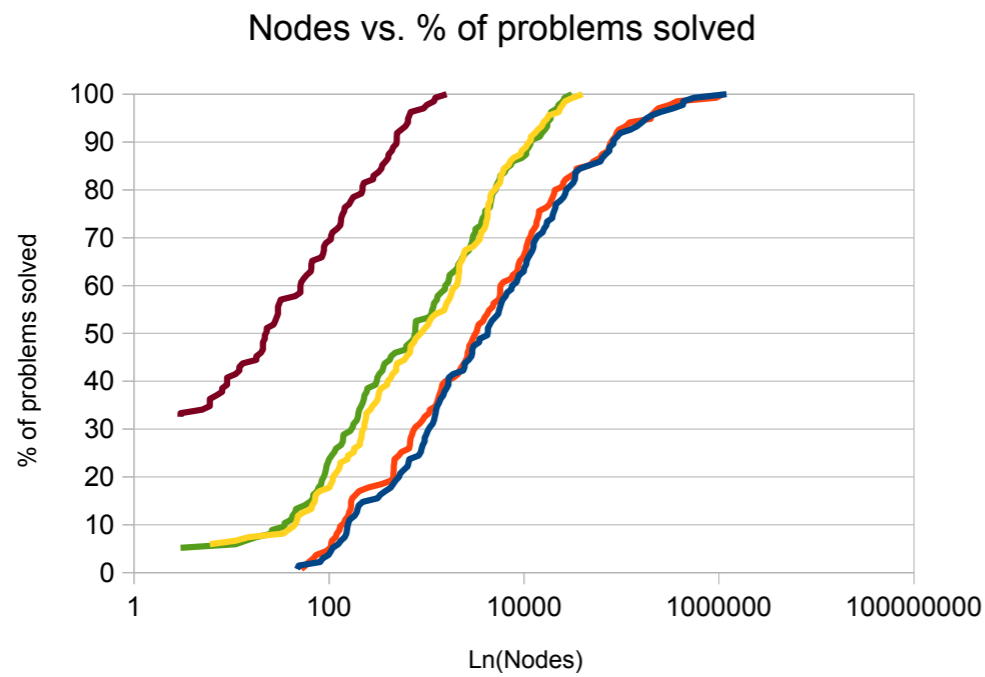
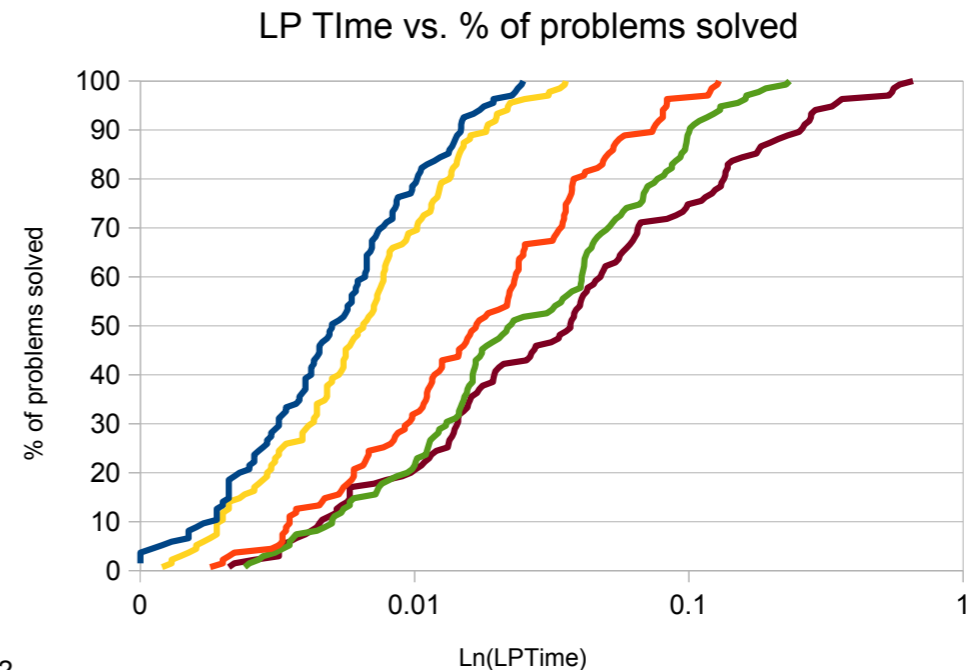
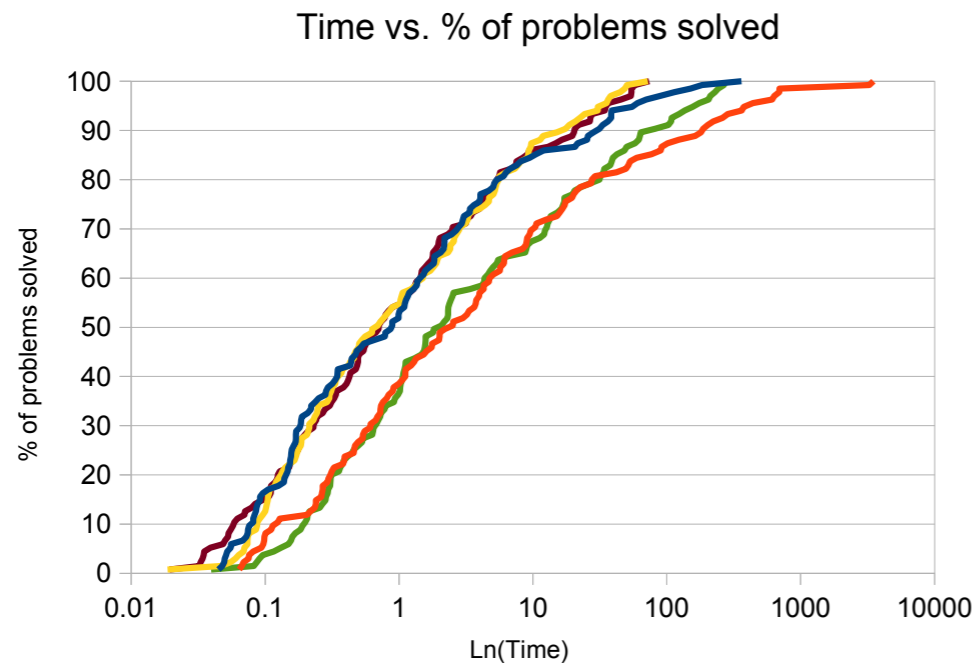
MIP3

$$\begin{aligned} \text{(MIP-}p\text{-G)} \quad & \max_{x,q} && \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \pi^k R_{ij}^k z_{ij}^k \\ \text{s.t.} & && \sum_{i \in I} \sum_{j \in J} z_{ij}^k = 1, && \forall k \in K \\ & && \sum_{i \in I} (C_{ij}^k - C_{il}^k) z_{ij}^k \geq 0 && \forall j, l \in J, \forall k \in K, \\ & && z_{ij}^k \geq 0 && \forall i \in I, \forall j \in J, \forall k \in K, \\ & && \sum_{i \in I} z_{ij}^k \in \{0, 1\} && \forall j \in J, \forall k \in K, \\ & && \sum_{j \in J} z_{ij}^k = \sum_{j \in J} z_{ij}^1 && \forall i \in I, \forall k \in K. \end{aligned}$$

About those MIPs

- $z_{LP}(MIP3) \leq z_{LP}(MIP2) \leq z_{LP}(MIP1)$
- $LP(MIP3)$ is “integer” for $k = 1$

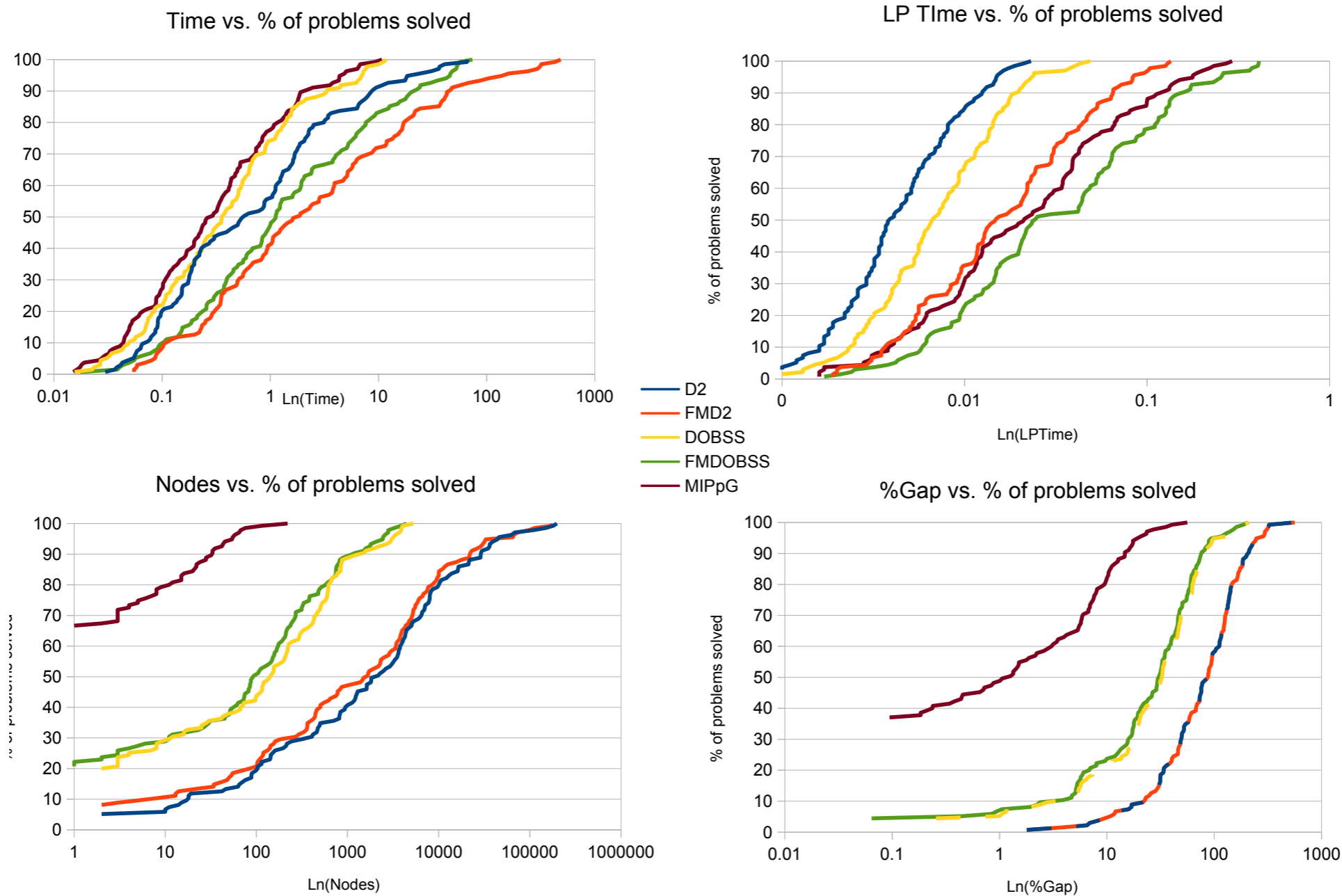
Computational comparison



- D2
- FMD2
- DOBSS
- FMDOBSS
- MIPpG

GSGs: $I=\{10,20,30\}$, $J=\{10,20,30\}$, $K=\{2,4,6\}$ —without variability.

Computational comparison



GSGs: $I=\{10,20,30\}$, $J=\{10,20,30\}$, $K=\{2,4,6\}$ —with variability.

	(D2)	(FMD2)	(DOBSS)	(FMDOBSS)	(MIP- <i>p</i> -G)
Mean Gap %	110.56	110.56	31.88	30.64	7.56

Stackelberg security game



- Payoffs depend only on which target is attacked and whether it is covered or not

	Covered	Uncovered
• Defender	$D^k(j c)$	$D^k(j u)$
Attacker	$A^k(j c)$	$A^k(j u)$

Compact representation of Stackelberg Security Games

- Resources-Targets settings can be modeled as a Stackelberg Game BUT if m resources and n targets then $\binom{n}{m}$ pure strategies!
- Stackelberg Security Games can be more compactly represented.
- Solve for optimal coverage probabilities of the targets.

Stackelberg security game:

“extended formulation”

$$\begin{aligned}
 (\text{QUAD}) \quad & \max_{x, q, a} \sum_{k \in K} \pi^k \sum_{j \in J} q_j^k (D^k(j|c) \sum_{i \in I: j \in i} x_i + D^k(j|u) \sum_{i \in I: j \notin i} x_i) \\
 \text{s.t.} \quad & \sum_{i \in I} x_i = 1, \\
 & \sum_{j \in J} q_j^k = 1 \quad \forall k \in K, \\
 & 0 \leq a^k - (A^k(j|c) \sum_{i \in I: j \in i} x_i + A^k(j|u) \sum_{i \in I: j \notin i} x_i) \leq (1 - q_j^k)M \quad \forall j \in J, \forall k \in K, \\
 & x_i \in [0, 1] \quad \forall i \in I, \\
 & q_j^k \in \{0, 1\} \quad \forall j \in J, \forall k \in K, \\
 & a^k \in \mathbb{R} \quad \forall k \in K.
 \end{aligned}$$

Stackelberg security game:

“extended formulation”

(QUAD) $\max_{x,q,a} \sum_{k \in K} \pi^k \sum_{j \in J} q_j^k (D^k(j|c) \sum_{i \in I: j \in i} x_i + D^k(j|u) \sum_{i \in I: j \notin i} x_i)$

s.t. $\sum_{i \in I} x_i = 1,$

$\sum_{j \in J} q_j^k = 1$

$0 \leq a^k - (A^k(j|c) \sum_{i \in I: j \in i} x_i + A^k(j|u) \sum_{i \in I: j \notin i} x_i) \leq (1 - q_j^k)M \quad \forall j \in J, \forall k \in K,$

$x_i \in [0, 1] \quad \forall i \in I,$

$q_j^k \in \{0, 1\} \quad \forall j \in J, \forall k \in K,$

$a^k \in \mathbb{R} \quad \forall k \in K.$

Stackelberg security game:

compact formulation

$$\begin{aligned}
 \text{(QUAD)} \quad & \max_{x, q, a} \sum_{k \in K} \pi^k \sum_{j \in J} q_j^k (D^k(j|c)c_j + D^k(j|u)(1 - c_j)) \\
 \text{s.t.} \quad & \sum_{i \in I} x_i = 1, \\
 & \sum_{i: j \in i} x_i = c_j \quad \forall j \in J, \\
 & x_i \in [0, 1] \quad \forall i \in I, \\
 & \sum_{j \in J} q_j^k = 1 \quad \forall k \in K, \\
 & 0 \leq a^k - (A^k(j|c)c_j + A^k(j|u)(1 - c_j)) \leq (1 - q_j^k)M \quad \forall j \in J, \forall k \in K, \\
 & q_j^k \in \{0, 1\} \quad \forall j \in J, \forall k \in K, \\
 & a^k \in \mathbb{R} \quad \forall k \in K.
 \end{aligned}$$

Stackelberg security game

compact formulation

(SECU-K-Quad) $\text{Max}_c \sum_{k \in K} \sum_{j \in J} p_k (q_j^k (c_j D^k(j|c) + (1 - c_j) D^k(j|u)))$

s.t. $c_j \in [0, 1] \quad \forall j \in J$

$\sum_{j \in J} c_j \leq m,$

$q_j^k (c_j A^k(j|c) - (1 - c_j) A^k(j|u)) \geq q_t^k (c_t A^k(t|c) - (1 - c_t) A^k(t|u)) \quad \forall k \in K,$

$q_j^k \in \{0, 1\} \quad \forall j \in J, \forall k \in K$

$\sum_{j \in J} q_j^k = 1 \quad \forall k \in K.$

Stackelberg security game



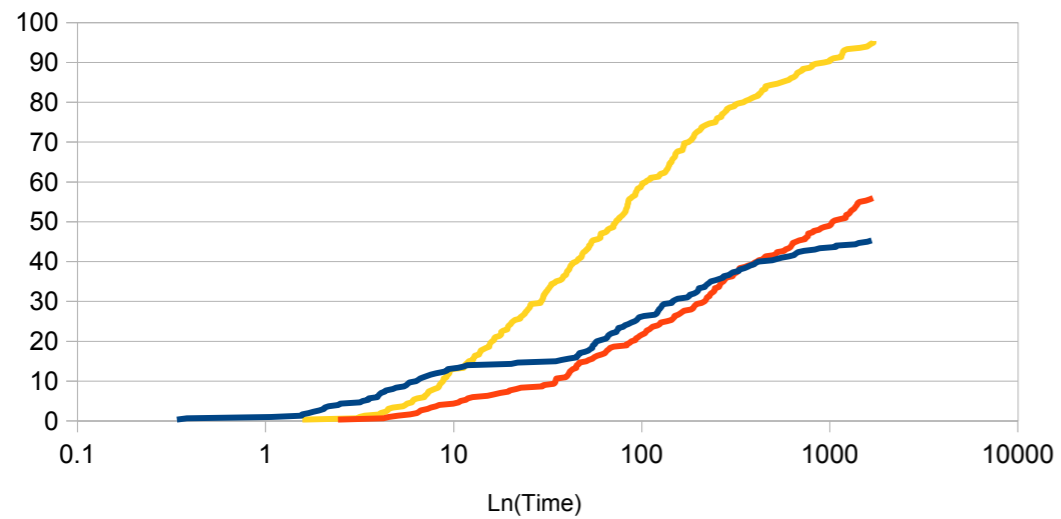
Stackelberg security game: MIP3-compact

$$\begin{aligned}
 (\text{SECU-p-MIP}) \quad & \text{Max}_y \quad \sum_{k \in K} \sum_{j \in J} p_k (D^k(j|c)y_{jj}^k + D^k(j|u)(q_j^k - y_{jj}^k)) \\
 \text{s.t.} \quad & \sum_{l \in J} y_{lj}^k \leq m q_j^k && \forall k, j, \\
 & 0 \leq y_{lj}^k \leq q_j^k, && \forall k, j \\
 & \sum_{j \in J} q_j^k = 1, && \forall k, \\
 & A^k(j|c)y_{jj}^k + A^k(j|u)(q_j^k - y_{jj}^k) - A(l|c)y_{lj}^k \\
 & \quad - A(l|u)(q_l^k - y_{lj}^k) \geq 0 && \forall j, l, k, \\
 & \sum_{l \in J} y_{lj}^k \in \{0, 1\} && \forall l, k, \\
 & \sum_{j \in J} y_{lj}^k = \sum_{j \in J} y_{lj}^1 && \forall l, k.
 \end{aligned}$$

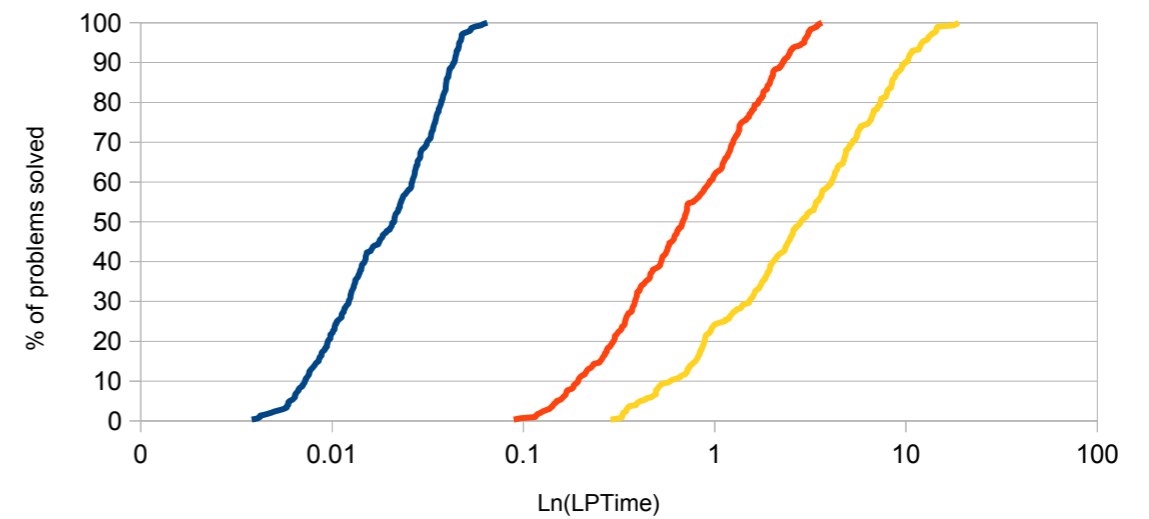
Link between MIP3 and SECU-p-MIP

- A pure strategy of the defender is a set of at most m targets
- $y_{hj}^k = \sum_{i \in I: h \in i} z_{ij}^k$
- $Proj(LP(P_{MIP3})) \subset LP(P_{SECU-p-MIP})$

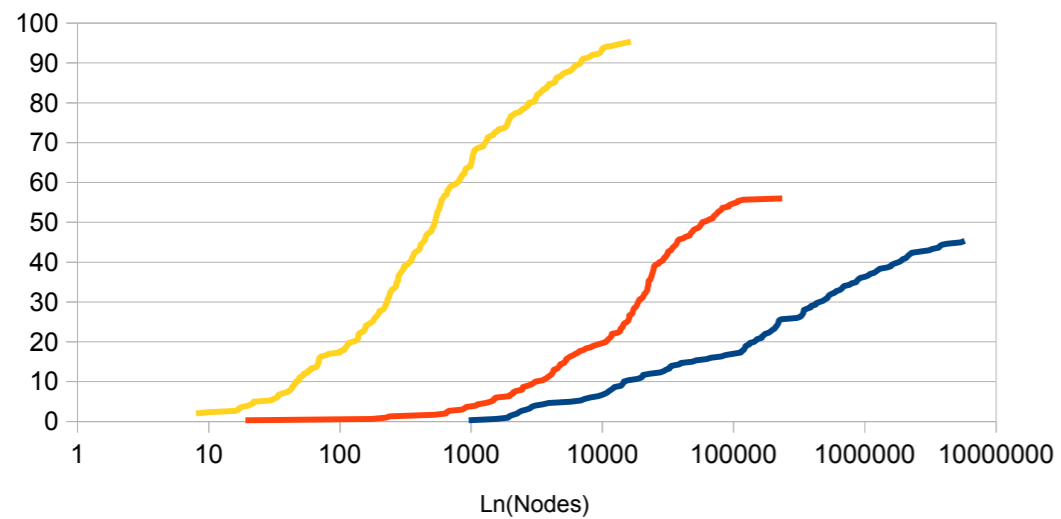
Time vs. % of problems solved



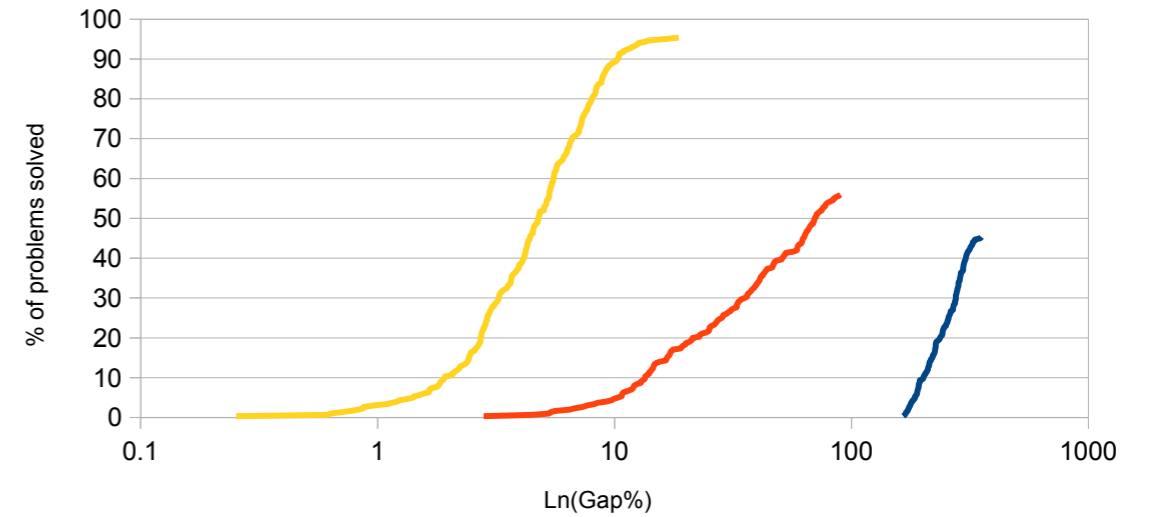
LP time vs. % of problems solved



Nodes vs. % of problems solved



Gap% vs. % of problems solved



— ERASER
— SDOBSS
— MIPpS

SSGs: $|K| \in \{4, 6, 8, 12\}$, $|J| \in \{30, 40, 50, 60, 70\}$, $m \in \{0.25|J|, 0.50|J|, 0.75|J|\}$

	(ERASER)	(SDBOSS)	(MIP- <i>p</i> -S)
Mean Gap %	204.82	28.76	1.72

Conclusions

- Bilevel models and MIP reformulations are appropriate to solve Stackelberg bimatrix games
- New MIP formulations for general and security cases
- A valid formulation is not enough!
- Future: develop decomposition solution approach (DW, Benders) based on strongest model.
- Future: study problems with non homogeneous resources, different second level...



How to determine mixed strategies from coverage probabilities

$$\sum_{h:j \in h} x_h = c_j, \forall j$$

$$\sum_h x_h = 1$$

$$x_h \geq 0, \forall h$$

Example: $m=2$

$$C_1=0.7, C_2=0.8, C_3=0.5$$

